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A MODEL FOR ORDINAL NONHIERARCHICAL CLUSTER METHODS.(U)
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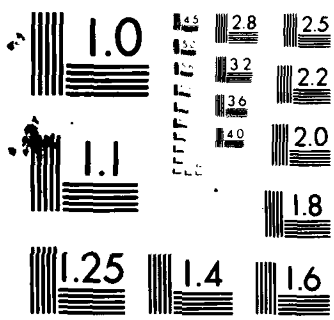

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A MODEL FOR ORDINAL NONHIERARCHICAL CLUSTER METHODS

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§1. Introduction. Hierarchical clustering involves the creation of a nested sequence of partitions of a set, whereas nonhierarchical clustering deals with a single partition. Nonhierarchical cluster techniques come up in a variety of situations involving such items as: (i) the assignment of stars to galaxies, (ii) the assignment of grades to students by an instructor; (iii) the classification of psychiatric patients by diagnostic type; (iv) the classification of rock formations: (v) the interpretation of LANDSAT images; (vi) automatic target detection; (vii) the grouping of companies according to properties that their stock might have in the stock market. It is not our purpose here to survey the literature in this area - rather, the reader is referred to such standard references as [1], [3], or [4].

As in our earlier model for hierarchical clustering [7], the viewpoint here will be that the input data has only ordinal significance. The order theoretic properties of the resulting model will be studied in §2, and it will be related to some earlier theoretical work ([5], [6]) in §3. Finally, §4 presents an abstract characterization of the model in terms of semiBoolean algebras.

The model is of course of some interest from an applied viewpoint because it provides a framework in which properties of at

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least some nonhierarchical cluster techniques might be developed. From a theoretical point of view, it leads to a class of semiBoolean algebra that has not been previously investigated, and suggests that the concept of SM-semilattice [5] might be of more importance than was previously suspected.

§2. The model. The input data for a nonhierarchical cluster method may be viewed as a finite nonempty set X together with a finite collection of attributes on X. Here an attribute will be viewed as a mapping from X into a partially ordered set A_i, where A_i might be the real number system, the nonnegative integers or some other partially ordered set. It will be assumed that the objects of X may be distinguished in terms of their attributes, so one can think of the input data as being a matrix (a_{ij}) where a_{ij} is the value of the jth attribute on the ith object. Thus if $X = \{x_1, x_2, \ldots, x_n\}$, then x_i may be identified with the vector $(a_{i1}, a_{i2}, \ldots, a_{in})$ of its attributes. A typical nonhierarchical cluster method involves the definition of some sort of optimality criterion that measures goodness of fit of a partition of X to the input data. An initial partition is formed and this is iteratively improved until the optimality criterion has exceeded some threshold value. Descriptions of such techniques may be found in any of the standard references on cluster analysis (see, for example, [1], [3], or [8]). It will be assumed that a rule has been given that allows one to associate with any subset Y of X values for each of the attributes. Such a rule might involve assigning to Y the mean of the values of an attribute for its members; other rules

might involve using the median, the mode or some other function of its values. Suppose then that a partition of X has classes $P_1, P_2, \ldots, P_k$, and that values have been assigned for each attribute to each class. The product ordering of the attribute sets A_i may then be used to impose a partial order on these classes. The result is an object called a partially ordered partition.

Definition 1. A partially ordered partition of a set X is a pair $(P, \leq)$ where P is a partition of X and $\leq$ is a partial order relation on the classes of P. The pair $(P, \leq)$ will be called an ordered partition if the relation $\leq$ is in fact a linear ordering.

The remainder of the paper will be concerned with ordered partitions. Such a situation might arise, for example, when the input data consists of a set X together with a single real-valued attribute. In such a situation, a cluster method may be viewed as a transformation of an ordered partition $(P, \leq)$ on X. Before one can reasonably investigate properties of such cluster methods, one must first investigate the structure of the set of all ordered partitions on a finite set X. Let $\underline{P} = (P, \leq)$ be a typical such partition. It will be convenient to let $[x]$ or x/P denote the unique class of P that contains the element x, and to specify the order relation $\leq$ by simply writing the lowest level class first, the next lowest class second,...,etc. To specify all member of a class, notation such as $(x_i x_j)$ or (x_i, x_j) will be useful. For J a proper subset of X, we will write $C(J)$ to denote the ordered partition $(X \backslash J)(J)$; thus

$C(J)$ has 2 classes $(X\backslash J)$ and J, with $(X\backslash J) \le J$. Finally, let $OP(n)$ denote the set of all ordered partitions on $\{1,2,\ldots,n\}$. Partially order $OP(n)$ be the rule

$(P_1, \le_1) \le (P_2, \le_2)$ if P_1 is a refinement of P_2, and

if $x/P_1 \to x/P_2$ preserves any existing finite joins and meets.

Thus in $OP(5)$, $(12)(34)(5) \le (12)(345)$, but $(12)(34)(5)$ $(345)(12)$.

The theory of ordered partitions may be viewed in at least 2 other ways. Let $X = \{1,2,\ldots,n\}$, and note that a linear order relation $\le$ on X produces an atom $(A, \le)$ in $OP(n)$. To say that $(A, \le) \le (P, \le')$ is clearly equivalent to the assertion that the equivalence relation associated with P is a congruence relation on $(X, \le)$. The ordering $OP(n)$ agrees with the usual ordering of congruence relations. However, in the study of congruences relations, there is a fixed linear order given on X, whereas in $OP(n)$, we have an ordered partition, and can ask such questions as "For which possible linear orders on X is it true that the given partition is a congruence relation?" Alternately, one can note that corresponding to any ordered partition $(P, \le)$ there is a relation S on X defined by xSy iff $[x] < [y]$. Thus $OP(n)$ is isomorphic to the set of all relations S such that:

(i) S is reflexive,

(ii) S is transitive,

(iii) S is total in that for $x,y \in X$, one must have xSy, ySx (or both).

The partial order on these relations is that of implication; i.e., $R \leq S$ if xRy implies xSy.

The remainder of this section will be devoted to the development of the order thoretic and combinatorial properties of $OP(n)$.

(P1) $OP(n)$ has $n!$ minimal elements (called atoms) and these are precisely the ordered partitions of the form $(\pi(1))(\pi(2))\ldots(\pi(n))$, where π is a permutation of $\{1,2,\ldots,n\}$.

Proof: The minimal elements are simply those ordered partitions whose classes are singleton subsets of $\{1,2,\ldots,n\}$.

(P2) For every atom $\underline{A}$, the interval above $\underline{A}$ is isomorphic to the lattice of all subsets of an $n-1$ element set.

Proof: There is no loss in generality in assuming that $\underline{A} = (1)(2)\ldots(n)$. The ordered partitions containing $\underline{A}$ are those of the form

$$(12\ldots j_1)(1+j_1,\ldots,j_2)\ldots(1+j_t,\ldots,n).$$

These are precisely those partitions that can be formed by removing pairs)(of parentheses from $(1)(2)\ldots(n)$. The proof is completed by noting that for $\underline{P}$, $\underline{Q}$ containing $\underline{A}$, the assertion $\underline{P} \leq \underline{Q}$ is equivalent to the assertion that the set of removed pairs of parentheses that produced $\underline{P}$ contains the set that produced $\underline{Q}$.

(P3) Every subset of OP(n) having a lower bound has a meet; <u>furthermore</u>, OP(n) <u>is a join semilattice</u>.

<u>Proof</u>: Let M be a subset of OP(n). If M has a lower bound, then there is an atom $\underline{A}$ under M. By P2, M has a meet. The fact that OP(n) is a join semilattice now follows from this and the fact that OP(n) has a largest element (12...n).

(P4) <u>There are</u> $2^n - 2$ <u>coatoms in</u> OP(n). <u>They consist of the ordered partitions having exactly two classes</u>.

<u>Proof</u>: This follows from the proof of P2.

(P5) OP(n) <u>is atomistic and dually atomistic</u>.

<u>Proof</u>: That OP(n) is dually atomistic is a consequence of P2. To see that it is atomistic, let $\underline{C} < \underline{D}$ in OP(n). We must show that there is an atom $\underline{A}$ under $\underline{D}$ that fails to be under $\underline{C}$. There is no loss in generality in assuming that (1)(2)...(n) is under $\underline{C}$. Suppose S,T are distinct classes of $\underline{C}$ that are merged by $\underline{D}$. Then if $s \in S$, $t \in T$, let π be the permutation of $\{1,2,\ldots,n\}$ that interchanges s and t, leaving all other numbers fixed. The desired atom is clearly $(\pi(1))(\pi(2))\ldots(\pi(n))$.

(P6) <u>Say that a coatom is of type</u> i <u>if its largest class has</u> i <u>members</u>. <u>This produces</u> n-1 <u>distinct types of coatoms having the following properties</u>:

(a) A coatom $\underline{C}$ is of type i iff it $C \wedge B$ exists for exactly i type 1 coatoms B.

(b) Every atom is the meet of exactly $n-1$ coatoms - one of each type.

(c) Let J be a proper subset of $\{1,2,\ldots,n\}$. Then $C(J) \wedge C(j)$ exists for $j \in J$ and fails to exist for $j \notin J$.

(d) Let J,K be subsets of $\{1,2,\ldots,n\}$. Then $C(J) \wedge C(K)$ exists iff $J \subseteq K$, or $K \subseteq J$. Indeed, if $J \subset K$, then $C(J) \wedge C(K) = (X\backslash K)(K\backslash J)(J)$, where $X = \{1,2,\ldots,n\}$.

(e) Let $\underline{C}_1,\underline{C}_2,\ldots,\underline{C}_t$ be a family of coatoms of $OP(n)$. If $\wedge_k \underline{C}_k$ fails to exist, there must exist a pair i,j of indices such $\underline{C}_i \wedge \underline{C}_j$ fails to exist.

Proof: (c) Suppose $C(J) \wedge C(j)$ exists, and let $(A, \leq_1)$ be an atom under $C(J) \wedge C(j)$. If $j \notin J$, then for $j_1 \in J$, we have $j <_1 j_1$ because $[j] > [j_1]$ in $C(j)$ and $j <_1 j_1$ because $[j] < [j_1]$ in $C(J)$. This contradiction shows that $j \in J$. On the other hand, if $j \in J$, then routine computation produces $C(J) \wedge C(j) = (X\backslash J)(J\backslash j)(j)$.

(d) Suppose $C(J) \wedge C(K)$ exists with $J \neq K$. We may assume that there is an element $k \in K\backslash J$. Suppose also that we can find an element j in $J\backslash K$. Then if $(A, \leq_1)$ is an atom under $C(J) \wedge C(K)$, we have $j <_1 k$ because of the ordering of $C(K)$ and $k <_1 j$ from the ordering of $C(J)$. This contradiction shows that $J \subset K$. Conversely, if $J \subset K$, then $C(J) \wedge C(K) = (X\backslash K)(K\backslash J)(J)$ follows

by a slight extension of the argument given in (c).

(a) and (b) are immediate consequences of (c) and (d).

(e) Assume that $\wedge_k \underline{C}_k$ does not exist, but that every pair $\underline{C}_i, \underline{C}_j$ has a meet. The $\underline{C}_k$'s must then be of different types, so we may arrange them in ascending order of types, relabelling if necessary. Assuming that $\underline{C}_i = C(J_i)$, we have from (d) that $J_1 \subset J_2 \subset \ldots \subset J_t$. Then $(X\backslash J_t)(J_t\backslash J_{t-1})\ldots(J_2\backslash J_1)(J_1) = \wedge_k \underline{C}_k$.

(P7) <u>There are</u> $\binom{n}{i}$ <u>coatoms of type</u> i <u>and each such coatom dominates exactly</u> $i!(n-1)!$ <u>atoms</u>. <u>In fact the interval under a type</u> i <u>coatom is isomorphic to</u> $OP(i) \times OP(n-i)$.

<u>Proof</u>: The assertion regarding the number of type i coatoms is a consequence of P6(c). Here is a sketch of the proof of the remaining assertions. Consider the coatom $C(J)$, where J has i elements. If $(\mathcal{B}, \leq)$ is under $C(J)$, let $\mathcal{B}_1$ be the restriction of $\mathcal{B}$ to $X\backslash J$ with $\mathcal{B}_2$ its restriction to J. Use the order imposed by $\leq$ to linearly order the classes of $\mathcal{B}_1$ and $\mathcal{B}_2$, and denote these orderings as $\leq_1$ and $\leq_2$. Then $(\mathcal{B}_1, \leq_1) \in OP(n-i)$, and $(\mathcal{B}_2, \leq_2) \in OP(i)$. The mapping $(\mathcal{B},\leq) \to ((\mathcal{B}_2,\leq_2),(\mathcal{B}_1,\leq_1))$ is easily shown to be the desired isomorphism. Now apply P1 to find the number of atoms under $C(J)$.

(P8) <u>The group of order automorphisms of</u> $OP(n)$ <u>is isomorphic to</u> $\underline{2} \times S_n$, <u>where</u> $\underline{2}$ <u>is a group of order</u> 2, <u>and</u> S_n <u>is the group of permutations on</u> $\{1,2,\ldots,n\}$.

<u>Proof</u>: For each $\pi \in S_n$, the correspondence $C(J) \to C(\pi(J))$ clearly extends to an order automorphism of $OP(n)$, and these order automorphisms are distinct. The correspondence $(B,\le) \to (B,\ge)$ is an order automorphism of order 2 that commutes with the order automorphisms induced by the elements of S_n. The subgroup of the group G of all order automorphisms of OP(n) generated by these mappings is thus clearly isomorphic to $\underline{2} \times S_n$. We would be done if we could just show that G has order $\le 2 \times n!$. By P7, an order automorphism of OP(n) must map a type 1 coatom into either a type 1 or a type n-1 coatom. Suppose that C(i) gets mapped to a type n-1 coatom by the order automorphism f. Choose $j,k \ne i$, and note that none of $fC(i) \wedge fC(j)$, $fC(i) \wedge fC(k)$, or $fC(j) \wedge fC(k)$ can exist. Since fC(i) is type n-1, it follows that at least one of fC(j) or fC(k) must also by type n-1. It then follows that the remaining one must be type n-1. Here we have used the fact that a type n-1 coatom has a meet with exactly 1 type 1 coatom. Thus if f sends one type 1 coatom to a type n-1 coatom, then it must send <u>all</u> type 1 coatoms to type n-1 coatoms. By P5, an order automorphism is completely determined by its effect on the coatoms of OP(n). There are only $2 \times n!$ ways that the n type 1 coatoms can be mapped onto either themselves or the type n-1 coatoms, and we are done. It should be noted that the result does indeed fail for OP(2); for OP(2) is a 3 element semilattice with 2 atoms and a unit element, so its group of automorphisms is simply $\underline{2}$.

(P9) For $\underline{A}$, $\underline{B} \in OP(n)$, there is a smallest element $\underline{D}$ over $\underline{A}$ such that $\underline{D} \wedge \underline{B}$ exists. This element $\underline{D}$ will be denoted $\underline{D} = \underline{A} \dotplus \underline{B}$.

Proof: Suppose $\underline{D}_1, \underline{D}_2 \geq \underline{A}$ and $\underline{D}_1 \wedge \underline{B}, \underline{D}_2 \wedge \underline{B}$ both exist. Since also $\underline{D}_1 \wedge \underline{D}_2$ must exist by P3, we may invoke P7(e) to see that $(\underline{D}_1 \wedge \underline{D}_2) \wedge \underline{B}$ exists.

We close this section by noting that since $OP(n)$ is dually atomistic, P7(e) extends easily to apply to families of elements that are not necessarily coatoms.

§3. Relation to SM-semilattices and SemiBoolean algebras. Before doing anything along these lines, some background material is needed. A mapping ϕ from a partially ordered set P to a partially ordered set Q is said to be residuated in case the preimage of a principal ideal of Q is necessarily a principal ideal of P; dually, ϕ is said to be residual if the preimage of a principal filter of Q is a principal filter of P. An alternate but illuminating definition for residuated mappings would state that $\phi: P \to Q$ is residuated if

(1) ϕ is isotone in that $a \leq b$ in P implies $\phi(a) \leq \phi(b)$ in Q, and there is an isotone mapping $\phi^+: Q \to P$ such that

(2) $p \leq \phi^+\phi(p)$ and $q \geq \phi\phi^+(q)$ for all $p \in P$, $q \in Q$.

The mapping ϕ^+ turns out to be residual and is completely determined by ϕ. Suppose now that P is a bounded meet semilattice and that Q has a smallest element 0. A residuated mapping $\phi: P \to Q$ is called semimultiplicative in case $\phi(a \wedge b) \neq 0$ implies that

$\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. To say that $\phi: P \to Q$ is range-closed is to say that its image is an order ideal of Q in that $q \leq \phi(a)$ implies the existence of an element p of P such that $q = \phi(p)$. For an introduction to the theory of residuated mappings, the reader might consult [2], while semimultiplicative residuated mappings were introduced in [5]. It will be convenient to let $SM(P,Q)$ denote the set of all semimultiplicative residuated mappings from P into Q, and denote this set as $SM(P)$ in case $P = Q$.

In the present paper we are dealing with partially ordered sets that have a largest but no smallest element. In such a context, it turns out to be useful to view the empty set as a principal ideal and say that $\phi: P \to Q$ is a partial residuated mapping in case the preimage of a principal ideal of Q in this extended sense is a principal ideal of P in the same sense. Such a mapping is either defined in all of P or on the complement of some ideal of the form $[\longleftarrow, p]$ for some suitable element p of P. To say that a partial residuated mapping is semimultiplicative will be to say that if $a \wedge b$ and $\phi(a \wedge b)$ are both defined, then $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ in Q. It will be convenient to denote the set of such mappings by $SMP(P,Q)$ or $SMP(P)$ in case $P = Q$. Notice that if P,Q do not have 0 elements, and if such elements are adjointed to them, then any partial residuated mapping ϕ from P into Q extends naturally to a residuated mapping from $P \cup \{0\}$ into $Q \cup \{0\}$ with $\phi \in SMP(P,Q)$ if its extension is a member of $SM(P,Q)$.

The notion of an SM-semilattice was introduced in [5], but the definition only applied to bounded semilattices. This is clearly not appropriate in the present context, so it is necessary to modify the definition as follows:

Definition. A partially ordered set P is said to be an SM-poset if for each $p \in P$, there is a range-closed idempotent member of SMP(P) whose image is $[\leftarrow, p]$. In view of [5], Theorem 2.2, p.433, this is equivalent to the following pair of conditions:

(i) Every principal filter of P is an implicative semilattice, and

(ii) For a given $p \in P$, it is either true that for every $x \in P$, there is a smallest element of the form $y \wedge p$ with $y \geq x$, or there is an element $p^* \in P$ with $M(p^*,p)$ and $p^* \wedge p$ not existing such that this condition is true for all $x \not\geq p^*$. (Note. $M(p^*,p)$ means that $b \leq p$ implies that $b = y \wedge p$ for some $y \geq p^*$).

In order to make contact with [5], Theorem 2,2, p. 433, we shall need

Lemma 1. For elements a,b of the partially ordered set P consider the following conditions:

(i) There is a smallest element of the form $x \wedge b$ with $x \geq a$.

(ii) There is a smallest element y above a such that $y \wedge b$ exists.

Then (ii) implies (i). If P is a join semilattice and if every

<u>principal filter of</u> P <u>is a lattice, then also</u> (i) <u>implies</u> (ii).

<u>Proof</u>: (ii) $\Longrightarrow$ (i) is trivial.

(i) $\Longrightarrow$ (ii) Let p be the smallest element of the form $x \wedge b$ with $x \geq a$. Then $p \vee a \geq a$, and $(p \vee a) \wedge b$ exists. If $x \geq a$, and $x \wedge a$ exists, then $x \wedge b \geq p$ forces $x \geq p$, so $x \geq p \vee a$, and we are done.

It follows from P9 and the above lemma that

<u>Theorem</u> 2. OP(n) <u>is an</u> SM-<u>poset</u>.

<u>Remark</u> 3. Let P be a lattice with 0, and Q a meet semilattice with 0. Suppose every proper filter of P is distributive. A minor modification of the proof of [5], Lemma 3.1, p. 437 will then show that for $\phi \in SM(P,Q)$, either

(1) $a \wedge b \neq 0$ implies $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$, or

(2) There is an element $w \in Q$ such that $w \leq \phi(a)$ for all $\phi(a) \neq 0$, and $\phi(a) \wedge \phi(b) = w$ whenever $\phi(a) \neq \phi(a \wedge b)$. It follows that the most general $\phi \in SM(P,Q)$ may be constructed as follows: Choose $w \in Q$ and $\bar{\phi} \in SM(P,Q\,[w,\longrightarrow])$ such that $a \wedge b \neq 0$ implies $\bar{\phi}(a \wedge b) = \bar{\phi}(a) \wedge \bar{\phi}(b)$. Let $p \in P$ be such that $\bar{\phi}(p) = w$. Now let $\phi = \bar{\phi}$ on the complement of $[0,p]$ with $\phi(x) = 0$ for $x \leq p$.

Suppose now that every proper principal filter of P and Q is in fact a Boolean algebra. Using the fact that residuated mappings preserve arbitrary existing suprema, it is easy to show that any

$\phi \in SM(P,Q)$ satisfies

(3) If $\wedge_i p_i$ exists in P and $\phi(\wedge_i p_i) \neq 0$, then $\phi(\wedge_i p_i) = \wedge_i \phi(p_i)$.

If also ϕ satisfies $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ whenever $a \wedge b \neq 0$, then it satisfies

(4) If $\wedge_i p_i \neq 0$, then $\phi(\wedge_i p_i) = \wedge_i \phi(p_i)$.

Remark 4. Let P be an SM-poset. Let $\phi \in SMP(P)$ be a range-closed idempotent mapping whose image is $[\leftarrow, b]$. If a,b have a lower bound in P, then by [5], Lemma 2.1, p.432, $\phi(a) = a \wedge b$. In particular, since ϕ^+ is defined for $\phi(a)$ with $\phi^+\phi(a), b$ having $\phi(a)$ as a lower bound, we have $\phi(a) = \phi\phi^+\phi(a) = \phi^+\phi(a) \wedge b$. If $x \geq a$ and $x \wedge b$ exists, then $\phi(x) = x \wedge b \geq \phi(a) = \phi^+\phi(a) \wedge b$. If every principal filter of P is a lattice, then by Lemma 1, $x \geq [\phi^+\phi(a) \wedge b] \vee a$. It follows that $[\phi^+\phi(a) \wedge b] \vee a = a \uparrow b$ is the smallest element above a which has a meet with b, and $a \uparrow b$ completely determines ϕ. If ϕ is defined on the complement of $[\leftarrow, p]$, then $p \leq \phi^+(b_1)$ for all $b_1 \leq b$ forces $M(p,b)$ in the sense of (ii) of the definition of SM-poset. Thus every such map ϕ is defined in the following manner: An element p of P is chosen such that $p \wedge b$ does not exist, $M(p,b)$, and such that $a \uparrow b$ exists whenever $a \nleq p$. Then $\phi(x) = (a \uparrow b) \wedge b$ for $x \nleq p$ with ϕ left undefined on $[\leftarrow, p]$. In particular, if $a \uparrow b$ exists for all a,b in P, then ϕ can be defined on all of P, and is completely determined by $a \uparrow b$. Since OP(n) has this property, this leads us to examine

the nature of the "arrow" operation $a \uparrow b$ when it is defined for all a,b in P.

Theorem 5. Let P be a join semilattice with no smallest element. Suppose that every filter of P is a distributive lattice, and that for every a,b in P, $a \uparrow b$ = the smallest element above a that has a meet with b exists. Then:

(1) $a \le a \uparrow b$.

(2) $a = a \uparrow b$ iff $a \wedge b$ exists.

(3) $a \le b$ implies $a \uparrow c \le b \uparrow c$.

(4) $b \le c$ implies $a \uparrow b \ge a \uparrow c$.

(5) If $a \wedge b$ exists, then $(a \wedge b) \uparrow c = (a \uparrow c) \wedge (b \uparrow c)$.

(6) $(a \vee b) \uparrow c = (a \uparrow c) \vee (b \uparrow c)$.

(7) $(a \uparrow b) \uparrow c = (a \uparrow b) \vee (a \uparrow c)$.

(8) If $b \wedge c$ exists, then $a \uparrow (b \wedge c) = (a \uparrow b) \vee (a \uparrow c)$.

Proof: The first 4 items are trivial.

(5) If $x \ge a \wedge b$ and $x \wedge c$ exists, then $x \vee a \ge a \uparrow c$ and $x \vee b \ge b \uparrow c$. Since $x = (x \vee a) \wedge (x \vee b)$, this shows that $x \ge (a \uparrow c) \vee (b \uparrow c)$. In particular, this is true for $x = (a \wedge b) \uparrow c$. The reverse inequality follows from (3).

(6) is clear.

(7) If $x \ge (a \uparrow b) \uparrow c$, then $x \ge a \uparrow b$ and $x \wedge c$ exists, so from $x \ge a$, we have also that $x \ge a \uparrow c$. Hence $x \ge (a \uparrow b) \vee (a \uparrow c)$. If $x \ge (a \uparrow b) \vee (a \uparrow c)$, then $x \ge a \uparrow b$ and $x \wedge c$ exists, so $x \ge (a \uparrow b) \uparrow c$.

(8) If $x \geq a$ and $x \wedge b \wedge c$ exists, then $x \wedge b$ and $x \wedge c$ must exist, so $x \geq (a \uparrow b) \vee (a \uparrow c)$. If $x \geq (a \uparrow b) \vee (a \uparrow c)$, then $x \geq a$ and $x \wedge b$, $x \wedge c$ exist. Since also $b \wedge c$ exists, we have that $x \wedge b \wedge c$ exists and $x \geq a \uparrow (b \wedge c)$.

§4. A characterization of OP(n). For a partially ordered set P, let us agree to call $a, b \in P$ related in case they have a common lower bound, and call them unrelated otherwise. We then have

Theorem 6. Let L be a poset having a largest element 1, but no smallest element. Suppose L satisfies the following conditions:

(1) L is atomic.

(2) For every atom p of L, $[p,1]$ is isomorphic to the lattice of all subsets of an n-1 element set.

(3) Among the coatoms of L there is a family S of n coatoms that is maximal with respect to being pairwise unrelated. Call these coatoms special, and suppose they are such that:

(3a) Corresponding to each proper subset J of S, there is a unique coatom $c(J)$ that is related to all $s \in J$ and unrelated to every $s \notin J$. Every coatom is of this form.

(3b) $\wedge_i c(J_i)$ exists iff the family $\{J_i\}$ are pairwise comparable as sets.

Then L is isomorphic to OP(n).

Proof: There are n special coatoms, so they may be labeled

$c(1), c(2), \ldots, c(n)$. Define $\theta: L \to OP(n)$ by $\theta(c(J)) = C(J)$, with $\theta(1)$ the unit element of $OP(n)$. For any other element x of L, by (2), x has a unique representation as $x = \wedge_i c(J_i)$ where there are at most n-1 coatoms $c(J_i)$. By (3b), the J_i's are pairwise comparable, so by property (P6), $\wedge_i C(J_i)$ exists in $OP(n)$. Define $\theta(x) = \wedge_i C(J_i)$. By the uniqueness of the representation of x as the meet of a family of coatoms, θ is well defined. To see that it is onto, note that if $D = \wedge_i C(J_i)$ exists in $OP(n)$, the argument we just made can be reversed to conclude that $\wedge_i c(J_i)$ exists in L. By construction, $\theta(\wedge_i c(J_i)) = \wedge_i C(J_i))$. The proof is completed by observing that $x \geq y$ in L is equivalent to the assertion that the set of coatoms above x be contained in the set above y, and this is clearly equivalent to $\theta(x) \geq \theta(y)$.

Though the above characterization is easy to establish, it is in many ways unsatisfactory. For one thing, conditions (3a) and (3b) are extremely powerful; for another, they are combinatorial as well as being order theoretic. It would be interesting to weaken these conditions, though as we shall soon see, they cannot be entirely eliminated. It would also be of interest to have a characterization that is more order theoretic. In that OP(n) is a semiBoolean algebra that happens also to be an SM-poset, a characterization along the lines of the triple construction for SM-semilattices [6] might be appropriate. It should be noted, however, that OP(n) is not triple constructible in the sense of [6], Theorem 15,

Before closing, it is illuminating to consider some examples. Each example is of a poset with height 3. Only the coatoms will be shown; a connection between a pair of coatoms will indicate that there is an atom beneath them. In that each atom will be under exactly two coatoms, this completely specifies the poset. Each example will have a set of 3 special coatoms. These will be indicated by closed circles, with the remaining coatoms denoted by open circles, Fig. 1 is the diagram for OP(3). Referring to the condition of Theorem 6, the example in Fig. 2 satisfies (1), (2), (3), (3a), but not (3b); the example in Fig. 3 saitsfies all conditions except (3a).

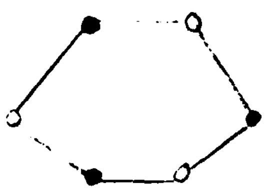

Fig. 1 OP(3)

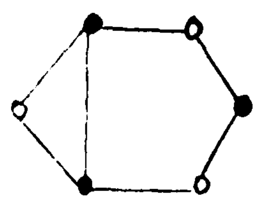

Fig. 2

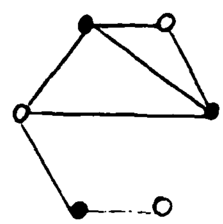

Fig. 3

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A model is presented for ordinal nonhierarchical clustering. The model involves the study of partitions whose classes are equipped with a linear order. Properties of the model are studied and a characterization is provided of the resulting structure.

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